Quadrangular Bézier Patches in Modeling and Modification of the Curvilinear Boundary Surface in 3D Problems of the Theory of Linear Elasticity in PIES

Eugeniusz Zieniuk, Krzysztof Szerszen
Faculty of Mathematics and Computer Science
University of Bialystok
Sosnowa 64, 15-887 Bialystok, Poland
Email: {ezieniuk, kszerszen}@ii.uwb.edu.pl

Abstract—**This paper proposes the use of quadrangular Bézier patches to model and modify the shape of the boundary for linear elasticity problems in 3D. The representation of the boundary in this way derives directly from computer graphics and have been analytically included in developed by the authors parametric integral equation systems (PIES). PIES are the modified classical boundary integral equations (BIE) in which the shape of the boundary can be modeled using a wide range of parametric curves and surfaces. The proposed approach eliminates the need for domain and boundary discretization in the process of solving boundary value problems, in contrast to popular traditional methods like FEM and BEM. On the basis of the proposed approach, a computer code has been written and examined through numerical examples.**

Index Terms—**parametric integral equation systems (PIES), boundary integral equations (BIE), Navier-Lame equations, linear elastics in 3D, quadrangular Bézier patches**

I. Introduction

Finite (FEM) and boundary (BEM) element methods in the process of solving boundary value problems replace continuous domain of governing equations by discrete mesh of finite [1,2,3,4] or boundary elements [5,6,7]. In order to improve the accuracy of results, it is usually necessary to break the domain into a larger number of smaller elements or use higher order elements. Additionally, numerical stability of solutions depends significantly on the adopted discretization scheme. A major focus of our research is the development a new method for solving boundary value problems based on proposed parametric integral equations systems (PIES), which in contrast to traditional methods (FEM, BEM) do not require domain or boundary discretization. The shape of the boundary in PIES is mathematically defined in the continuous way by means known from computer graphics parametric curves in the case 2D and parametric surfaces in the case of 3D boundary value problems, and its definition is practically reduced to giving a small set of control points. So far, however, PIES has been mainly used to solve 2D potential boundary value problems modeled by partial differential equations such as: Laplace [8], Helmholtz [9], Poisson [10], and Navier-Lame [11]. High accuracy and effectiveness of PIES for those problems has been encouraging its generalization to 3D boundary problems. In the case of 3D problems, we have restricted our considerations only for Laplace [12] and Helmholtz [13] equations. We have also begun preliminary work investigating the use this strategy for the Navier-Lame equations to solve 3D linear elastic problem in polygonal domains [14]. The aim of this paper is to explore the potential benefits of using quadrangular Bézier surface patches to model and modify the curvilinear shape of the boundary in 3D problem of linear elasticity modeled by Navier-Lame equations and solved by PIES. The formula of PIES for the Navier-Lame equations has already been obtained by adopting the general assumption that the boundary can be described in general way by any parametric functions. However, the assessment of efficiency of using Bézier patches for boundary representation requires direct substitution them into PIES. This has been done on the basis on written computer program whose results are presented in the last part of this article.

II. Qudrangular Bézier Patches

In obtained PIES for 2D boundary problems, the boundary geometry is considered in its mathematical formalism and defined with help of parametric linear segments [11] and also by parametric curves [8,9]: Bézier, B-spline and Hermite. The proposed strategy for 2D boundary description has been proved to be effective both from the point of view of simplifying the boundary declaration as well as improving the accuracy of obtained solutions compared with FEM and BEM. Boundary problems in 3D are characterized by much higher complexity. In this article we evaluate quadrangular Bézier surface patches as boundary representation in 3D. Quadrangular Bézier patches are an extension of parametric Bézier curves. Both curves and surfaces can be simply declared by a limited set of control points. Quadrangular Bézier surfaces are specified by an array of $n \times m$ control points P_{ij} and written mathematically by [15][16]

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$$P(v,w)=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{ij}B_{im}(v)B_{jn}(w),\ 0\le v,w\le 1, \qquad (1)$$

where $B_{im}(v)$, $B_{jn}(w)$ are Bernstein basis functions written in the form analogous like for Bézier curves

$$B_{im}(v)=\binom{m}{i}v^{i}(1-v)^{m-i},\ B_{jn}(w)=\binom{n}{j}w^{j}(1-w)^{n-j}. \qquad (2)$$

The main advantage of Bézier patches lies in the simplicity of their creation and modification with help of a limited number of control points. In this paper, we deal with Bézier patches for $n=m=1$ and $n=m=3$ (also referred here as degree of 1st and 3rd). Bézier patch of first degree takes the form of a quadrangular plate defined by four points $P_{00},P_{10},P_{11},P_{01}$ placed at the corners of defined rectangle as shown in Fig. 1.

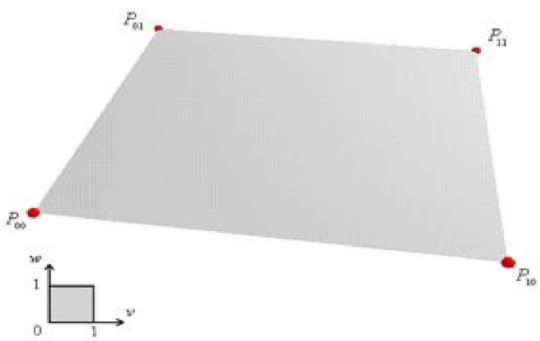

Figure 1. Quadrangular Bézier surface patch of degree 1 with 4 corner points.

The patch is described by following matrix form [17

$$P(v,w)=\begin{bmatrix}1-v & v\end{bmatrix}\begin{bmatrix}P_{00} & P_{01}\\ P_{10} & P_{11}\end{bmatrix}\begin{bmatrix}1-w\\ w\end{bmatrix},\ 0\le v,w\le 1 \qquad (3)$$

In turn, Bézier patch of degree 3 is defined by 16 control points. Its visualization is shown below.

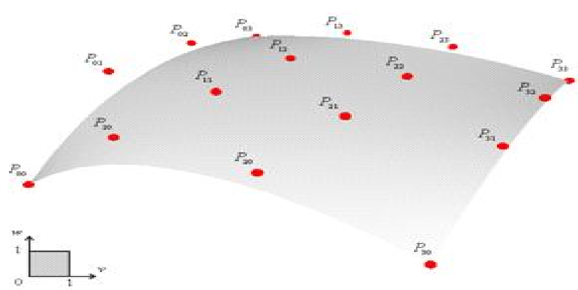

Figure 2. Quadrangular Bézier surface patch of degree 3 with 16 control points

The patch of degree 3 can be described by following matrix form [17]

$$P(v,w)=\begin{bmatrix}1 & v & v^2 & v^3\end{bmatrix}M^{T}\begin{bmatrix}P_{00} & P_{01} & P_{02} & P_{03}\\ P_{10} & P_{11} & P_{12} & P_{13}\\ P_{20} & P_{21} & P_{22} & P_{23}\\ P_{30} & P_{31} & P_{32} & P_{33}\end{bmatrix}M\begin{bmatrix}1\\ w\\ w^2\\ w^3\end{bmatrix}, \qquad (4)$$

$$M=\begin{bmatrix}1 & -3 & 3 & -1\\ 0 & 3 & -6 & 3\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 1\end{bmatrix},$$

Fig. 3 show sample modification of Bézier patch of the third degree by moving individual control points.

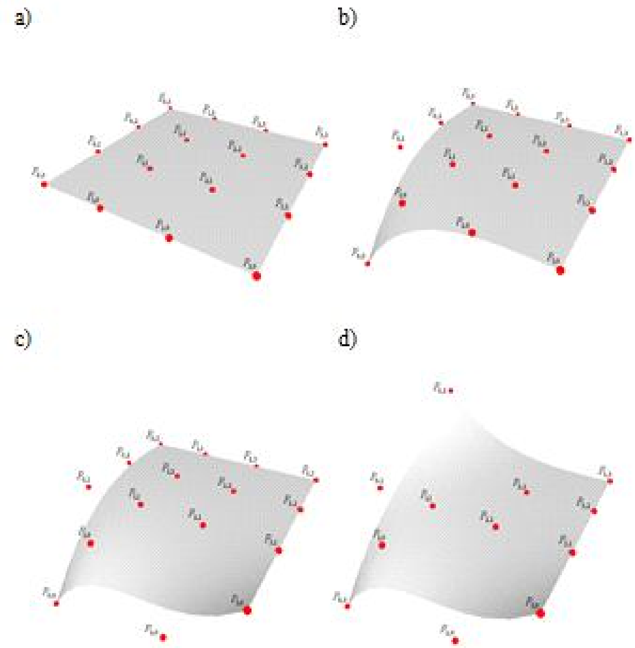

Figure 3. Examples of shapes defined with Bézier patch of degree 3: a) the initial flat surface, the updated surfaces after moving selected b) P_{00}, c) P_{20} and d) P_{03} control points.

III. Modeling And Modification Of The Curvilinear Boundary Surface By Control Points

A single Bézier patch can only capture a small class of shapes. The potential ability to model more complicated boundary shapes lies in the increase the degree of the polynomials used in the basis functions (2). From practical point of view it is connected with increase the number of used control points. However, the relative lack of local shape control by patches of higher degree makes them more difficult to accurate capture the real shapes. In practice, the most common solution is to join independent low degree patches together into a piecewise surface. Fig. 4 shows declaration of the boundary in PIES defined by five Bézier patches of first and the one of the third degree. The boundary is practically defined by 20 control points, of which 16 belong to curvilinear patch of degree 3. By appropriately selecting and moving one or more control points it becomes possible to effective change the shape of the declared boundary. In many inverse problems, concerning for example shape identification or optimization, it is necessary to perform multiply modifications

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of initial shape of the boundary. The declaration of boundary geometry on the basis of Bézier patches allows for easy modification of the boundary by moving individual or group of control points. Fig 4b shows the modification of the initial domain by moving only one P_{22} point.

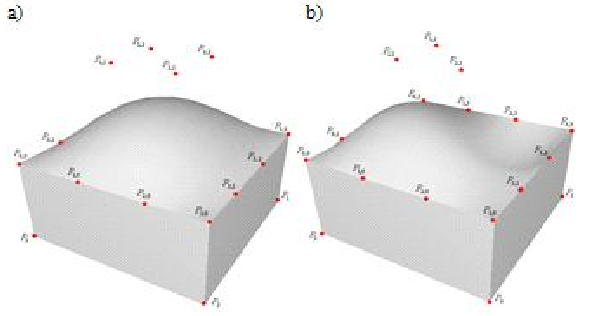

Figure. 4. Modeling the boundary by Bézier patches: a) posed control points, b) modification of initial shape by moving P_{22}.

Two variants of the geometry from Fig. 4 show the effectiveness and simplicity of proposed boundary shape modification performed using a small input data by only control points. Fig. 5, in turn, shows the further steps to create a part of a hollow cylinder with flat and curvilinear Bézier patches of degree 1 and 3 respectively.

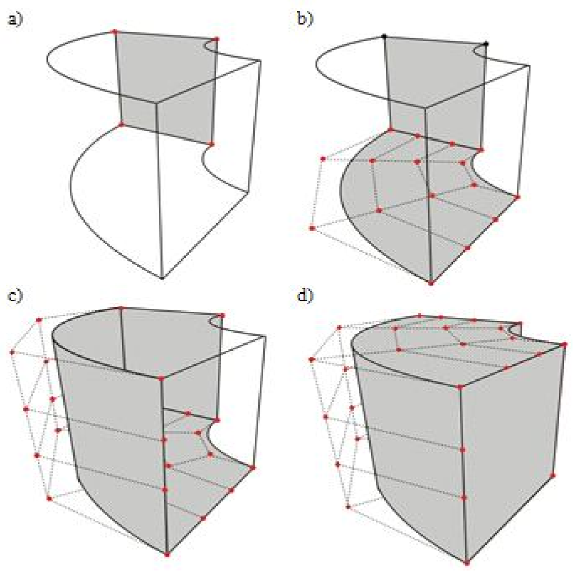

Figure. 5. Further stages of defining a part of a hollow cylinder bounded by Bézier patches: a) the patch of the first degree, b), c) patches of the third degree, d) the complete geometry.

The domain is described unequivocally and accurately only by 48 control points of the individual surface patches. *The calculation* required to find the locations of control points for Bézier patches of degree 3 from the example is presented in Fig. 6.

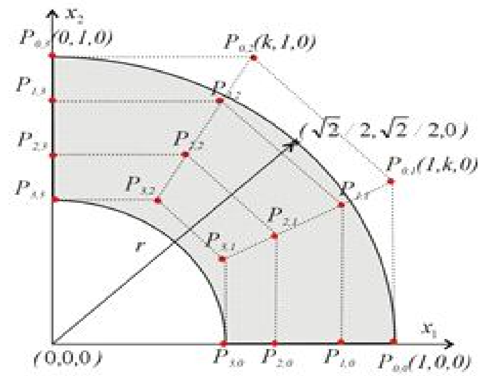

Figure. 6. Determination of the exact locations of the control points for Bézier patch from the base of the cylindrical domain.

The outer edge of the patch from Fig. 5 formed by the points $P_{00}, P_{01}, P_{02}, P_{03}$ should take the shape of a quarter circle of radius r. Assuming that the center point of the circle is $(0,0,0)$ and radius $r=1$ the coordinates of control points are as follows $P_{00}(1,0,0)$, $P_{01}(1,k,0)$, $P_{02}(k,1,0)$, $P_{03}(0,1,0)$. The coefficient k may be determined by formula (5), where for $v=0$, $w=0.5$ the outer edge of the patch should pass through the point $(\sqrt{2}/2, \sqrt{2}/2, 0)$, hence

$$\sqrt{2}/2 = (1-0.5)^3 + 3k(1-0.5)^2 0.5 + 3(1-0.5)0.5^2 + 0.5^3 \quad (5)$$

and $k = 4(\sqrt{2}-1)/3$.

The location of other control points from boundary geometry from Fig. 5 are determined in the same way, taking into account the appropriate value of the radius. Presented boundary representation by Bézier patches is directly included in presented in the next section mathematical formalism of PIES. Therefore, modeling the boundary is only limited to the above-mentioned declaration of surface patches without their further division, for example into boundary or finite elements. This is a significant advantage of PIES, which leads to simplification of both the boundary geometry description and numerical calculations.

IV. PIES For The Navier-Lame Equation

In the previous section, we have shown that the shape of 3D domains can be modeled using parametric surface patches. This is very effective from the point of view of interactive modification and creation of 3D domains and also their visualization. Although in the boundary value problems we also describe the shape the domain and boundary, the main focus of our attention is not the visual representation of their geometry, but finding solutions on the boundary or in the domain for given boundary conditions, described shape of the domain and the introduced differential equations that are solved. This makes the problem much more complex. In boundary value problems solved using traditional numerical methods it is needed to divide computational continuous physical domain into elements. This approach requires a large

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amount of elements, but it is easy to apply to the classical boundary integral equations (BIE). BIE is characterized by the fact that the boundary is defined in general way by the boundary integral [18]. Therefore, BEM used to solve BIE requires discretization of the boundary. Such boundary representation has also its disadvantages, because any modification of the boundary shape requires a re-discretization of the boundary. Unfortunately, regardless of its efficiency presented in the previous section the way of modeling and modification of the curvilinear boundary surface by parametric patches is not possible to apply directly in classical BIE. Therefore an analytical modification of BIE have been performed. This modification has been performed on the basis of assumptions that the boundary is not defined using the boundary integral but with help of discussed in previous sections surface patches. PIES for the Navier-Lame equations in 3D has been obtained as a result of the analytical modification of the traditional BIE. Methodology of this modification for 2D problems was presented in [11]. Generalizing this modification for Navier-Lame equations in 3D, we obtain the following formula of PIES [14]

$$0.5\boldsymbol{u}_l(v_1,w_1)=\sum_{j=1}^{n}\int_{v_{j-1}}^{v_j}\int_{w_{j-1}}^{w_j}\{\overline{\boldsymbol{U}}_{lj}^{*}(v_1,w_1,v,w)\boldsymbol{p}_j(v,w) - \overline{\boldsymbol{P}}_{lj}^{*}(v_1,w_1,v,w)\boldsymbol{u}_j(v,w)\}J_j(v,w)dvdw, \quad (6)$$

where $v_{l-1}<v_1<v_l$, $w_{l-1}<w_1<w_l$, $v_{j-1}<v<v_j$, $w_{j-1}<w<w_j$, $l=1,2,3...,n$, and n is the number of parametric patches that create the domain boundary in 3D. These patches is included in the kernels (integrand functions) of the obtained PIES. Therefore PIES is not defined on the boundary as BIE, but on the mapping plane. Such a definition of PIES has the advantage that the change of the boundary is automatically adjust of PIES to the modified shape without any additional re-discretization as in the case of classical BEM. In this paper the patches takes form of Bézier surfaces that create closed 3D domain.

The integrands $\overline{\boldsymbol{U}}_{lj}^{*}$, $\overline{\boldsymbol{P}}_{lj}^{*}$ in equation (6) are represented in the following matrix form

$$\overline{\boldsymbol{U}}_{lj}^{*}(v_1,w_1,v,w)=\frac{(1+\nu)}{8\pi(1-\nu)E\eta}\begin{bmatrix}U_{11} & U_{12} & U_{13}\\ U_{21} & U_{22} & U_{23}\\ U_{31} & U_{32} & U_{32}\end{bmatrix}, \quad (7)$$

$$\overline{\boldsymbol{P}}_{lj}^{*}(v_1,w_1,v,w)=\frac{-1}{8\pi(1-\nu)\eta^2}\begin{bmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{bmatrix}. \quad (8)$$

The individual elements in the matrix (7) can be expressed in an explicit form as

$$U_{11}=(3-4\nu)+\frac{\eta_1^2}{\eta^2},\; U_{12}=\frac{\eta_1\eta_2}{\eta^2},\; U_{13}=\frac{\eta_1\eta_3}{\eta^2},$$

$$U_{21}=\frac{\eta_2\eta_1}{\eta^2},\; U_{22}=(3-4\nu)+\frac{\eta_2^2}{\eta^2},\; U_{23}=\frac{\eta_2\eta_3}{\eta^2},$$

$$U_{31}=\frac{\eta_3\eta_1}{\eta^2},\; U_{32}=\frac{\eta_3\eta_2}{\eta^2},\; U_{33}=(3-4\nu)+\frac{\eta_3^2}{\eta^2},$$

while in the matrix (8) by

$$P_{11}=((1-2\nu)+3\frac{\eta_1^2}{\eta^2})\frac{\partial\eta}{\partial n},$$

$$P_{12}=3\frac{\eta_1\eta_2}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_1 n_2-\eta_2 n_1}{\eta},$$

$$P_{13}=3\frac{\eta_1\eta_3}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_1 n_3-\eta_3 n_1}{\eta},$$

$$P_{21}=3\frac{\eta_2\eta_1}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_2 n_1-\eta_1 n_2}{\eta},$$

$$P_{22}=((1-2\nu)+3\frac{\eta_2^2}{\eta^2})\frac{\partial\eta}{\partial n},$$

$$P_{23}=3\frac{\eta_2\eta_3}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_2 n_3-\eta_3 n_2}{\eta},$$

$$P_{31}=3\frac{\eta_3\eta_1}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_3 n_1-\eta_1 n_3}{\eta},$$

$$P_{32}=3\frac{\eta_3\eta_2}{\eta^2}\frac{\partial\eta}{\partial n}-(1-2\nu)\frac{\eta_3 n_2-\eta_2 n_3}{\eta},$$

$$P_{33}=((1-2\nu)+3\frac{\eta_3^2}{\eta^2})\frac{\partial\eta}{\partial n}.$$

Created by Bézier patches graphical representation of the 3D domain is directly incorporated into the mathematical formula of PIES without requiring an additional its division, for example into boundary elements. Kernels (7,8) include in its mathematical formalism the shape of a closed boundary, created by means of appropriate relationships between Bézier patches $(l, j=1,2,3...,n)$, which are defined in Cartesian coordinates using the following relations

$\eta_1=P_j^{(1)}(v,w)-P_l^{(1)}(v_1,w_1)$, $\eta_2=P_j^{(2)}(v,w)-P_l^{(2)}(v_1,w_1)$ (4)

$\eta_3=P_j^{(3)}(v,w)-P_l^{(3)}(v_1,w_1)$ and $\eta=\left[\eta_1^2+\eta_2^2+\eta_3^2\right]^{0.5}$ (9)

where $P_j^{(1)}(v,w)$, $P_j^{(2)}(v,w)$, $P_j^{(3)}(v,w)$ are the scalar components of Bézier patches, described by formula (1). This notation is also valid for the patch labeled by index l with parameters $v_1\, w_1$, ie. for $j=l$ and for parameters $v=v_1$ and $w=w_1$. It is possible to insert into (9) Bézier patches of any degree expressed by (1). In the paper the patches of degree 1 and 3 is used.

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The smooth character of Bézier surfaces requires new ways about computing the Jacobian $J_j(v,w)$ and normal derivatives $n_1(v,w)$, $n_2(v,w)$, $n_3(v,w)$ in (7,8). In the case of Bézier patch of degree 1 the Jacobian and normal derivatives are constant on the whole surface area regardless of the actual values for parameters v, w. While in the case of the patch of degree 3 these parameters change at any point of the patch, depending on changes of v, w. A formula for the Jacobian and normal derivatives can be determined analytically from expression (1) that define quadrangular Bézier surfaces as follows

$$n_m(v,w) = \frac{A_m(v,w)}{J_j(v,w)}, \quad m = 1,2,3 \qquad (10)$$

where

$$A_1(v,w) = \frac{\partial P_l^{(2)}(v,w)}{\partial w}\frac{\partial P_l^{(3)}(v,w)}{\partial v} - \frac{\partial P_l^{(2)}(v,w)}{\partial v}\frac{\partial P_l^{(3)}(v,w)}{\partial w},$$

$$A_2(v,w) = \frac{\partial P_l^{(3)}(v,w)}{\partial w}\frac{\partial P_l^{(1)}(v,w)}{\partial v} - \frac{\partial P_l^{(3)}(v,w)}{\partial v}\frac{\partial P_l^{(1)}(v,w)}{\partial w},$$

$$A_3(v,w) = \frac{\partial P_l^{(1)}(v,w)}{\partial w}\frac{\partial P_l^{(2)}(v,w)}{\partial v} - \frac{\partial P_l^{(1)}(v,w)}{\partial v}\frac{\partial P_l^{(2)}(v,w)}{\partial w},$$

and

$$J_j(v,w) = [A_1^2(v,w) + A_2^2(v,w) + A_3^2(v,w)]^{0.5}.$$

The outward or inward direction of normal derivatives $n_1(v,w)$, $n_2(v,w)$, $n_3(v,w)$ is determined by the appropriate numeration of control points of Bézier patches. The use of PIES for solving 2D and 3D boundary problems made it possible to eliminate the need for discretization of both the mentioned above boundary geometry, and the boundary functions. In previous research works, the boundary functions both defined as the boundary conditions, as well as obtained after solving PIES are approximated by Chebyshev polynomials [8]. The numerical solution of PIES for 3D problems is very similar to that previously described for 2D problems [9]. The difference appears in the use of more complex series to approximate the solutions, which are presented in the following form

$$\boldsymbol{u}_j(v,w) = \sum_{p=0}^{N}\sum_{r=0}^{M} \boldsymbol{u}_j^{(pr)} T_j^{(p)}(v) T_j^{(r)}(w), \qquad (11)$$

$$\boldsymbol{p}_j(v,w) = \sum_{p=0}^{N}\sum_{r=0}^{M} \boldsymbol{p}_j^{(pr)} T_j^{(p)}(v) T_j^{(r)}(w), \qquad (12)$$

where $\boldsymbol{u}_j^{(pr)}, \boldsymbol{p}_j^{(pr)}$ are requested coefficients and $T_j^{(p)}(v)$, $T_j^{(r)}(w)$ are Chebyshev polynomials. One of these functions, either $\boldsymbol{u}_j(v,w)$ or $\boldsymbol{p}_j(v,w)$, depending on the type of the resolved boundary problem, is posed in the form of boundary conditions, whereas the other is the searched function resulting from the solution of PIES. Both the posed and searched boundary functions are approximated using the generalized approximating series (11) or (12). Finally, after substituting these series to formula (6) and writing it at the collocation points, we obtain an algebraic equation system with respect to the unknown coefficients $\boldsymbol{u}_j^{(pr)}$ or $\boldsymbol{p}_j^{(pr)}$.

V. Numerical Examples

In order to evaluate proposed PIES for defined by Bézier patches domains shown in Fig. 4a and 5d, a numerical verification was performed with displacement Dirichlet boundary conditions for the Navier-Lame equations obtained from equations (13). These formulas are also exact solutions of the Navier-Lame equations [19]

$$u_1(\boldsymbol{x}) = \frac{2x_1 + x_2 + x_3}{2}, \; u_2(\boldsymbol{x}) = \frac{x_1 + 2x_2 + x_3}{2}, \qquad (13)$$

$$u_3(\boldsymbol{x}) = \frac{x_1 + x_2 + 2x_3}{2}.$$

The normal directives of functions (13) computed on the surface of the Bézier patches can be identified with exact analytical solutions of the problem on the boundary. The derived normal derivatives for these functions are presented below

$$p_1(\boldsymbol{x}) = \frac{du_1(\boldsymbol{x})}{dn} = 2n_1(\boldsymbol{x}) + 0.4n_2(\boldsymbol{x}) + 0.4n_3(\boldsymbol{x}),$$

$$p_2(\boldsymbol{x}) = \frac{du_2(\boldsymbol{x})}{dn} = 0.4n_1(\boldsymbol{x}) + 2n_2(\boldsymbol{x}) + 0.4n_3(\boldsymbol{x}), \qquad (14)$$

$$p_3(\boldsymbol{x}) = \frac{du_3(\boldsymbol{x})}{dn} = 0.4n_1(\boldsymbol{x}) + 0.4n_2(\boldsymbol{x}) + 2n_3(\boldsymbol{x}).$$

Table 1 shows the accuracy of solutions obtained on the boundary formed by Bézier patch of degree 3 for the domain shown in Fig. 4. We have analyzed different shapes of this patch, modified by changing the position of four central control points $P_{11}, P_{12}, P_{21}, P_{22}$. We choose $E = 1Pa$ and $\nu = 0.25$ for all calculations.

Table. 1. The errors norm L_2 of solutions in PIES for various shapes of Bézier patch of degree 3 from Fig. 4.

Control points $P_{11}, P_{12}, P_{21}, P_{22}$	The errors norm L_2 $\|e\|_{p_1}$, $\|e\|_{p_2}$, $\|e\|_{p_3}$
P_{11}(0.33, 0.33, 1.25) P_{12} (0.33, 0.67, 1.25) P_{21} (0.67, 0.33, 1.25) P_{22} (0.67, 0.67, 1.25)	0.0492533 % 0.0507693 % 0.0959662 %
P_{11}(0.33, 0.33, 1.5) P_{12} (0.33, 0.67, 1.5) P_{21} (0.67, 0.33, 1.5) P_{22} (0.67, 0.67, 1.5)	0.100674 % 0.086942 % 0.119345 %
P_{11}(0.33, 0.33, 1.5) P_{12} (0.33, 0.67, 1.5) P_{21} (0.67, 0.33, 1.5) P_{22} (0.67, 0.67, 0.5)	0.193235 % 0.260095 % 0.152921 %

DOI: 01.IJCSI.03.01. 61

The error norms L_2 for solutions from column 2 are calculated on the basis of the following formula [20]

$$\| e \| = \frac{1}{\left|\overline{p^{(k)}}\right|_{\max}} \sqrt{\frac{1}{K} \sum_{k=1}^{K} \left(p^{(k)} - \overline{p^{(k)}} \right)^2} *100\%, \qquad (15)$$

where $p^{(k)}$ represents a set of $K = 85$ solutions in PIES obtained on the boundary represented by the patch placed on upper face of the cube, while $\overline{p^{(k)}}$ are exact solutions given by equations (14). The solutions on the boundary were obtained with $N = M = 9$ coefficients in the approximating series (12). Table 2, in turn, lists the errors obtained in PIES related to exact values (14) at selected points on Bézier patch of degree 3 which forms an outer cylindrical area in Fig. 5d.

Table. 2. The relative error of solutions at selected points on outer Bézier patch of degree 3 from Fig. 5d.

x_1, x_2, x_3	Relative error p_1, p_2, p_3 [%]
5.82562, 1.44187,1.39286	0.561040, 0.385302, 0.559118
3.16054, 5.10106, 1.39286	0.480317, 0.019696, 0.071019
5.52849, 2.33547, 1.39286	0.108074, 0.429155, 0.262718
2.33547, 5.52849, 0.39616	0.804894, 0.055514, 0.125395
1.44187,5.82562, 0.99251	0.848432, 0.000305, 0.889448

The error norms L_2 (15) for solutions obtained in 340 points on all six Bézier patches from geometry shown Fig. 5d for components p_1, p_2, p_3 take values 0.40159%, 0.415836% and 0.700875 % respectively. Low level of this errors demonstrate a sufficiently good agreement of solutions in PIES with exact solutions (14).

Conclusions

In this paper, our purpose is to discuss the possibility of using quadrangular Bézier surface patches with different degrees to describe 3D geometry for solving linear elastic problems in PIES. This approach in conjunction with PIES makes it possible to develop a completely non-element method for solving 3D boundary problems. Numerical example demonstrates the simplicity of the modeling procedure and the good accuracy of the results.

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